Planar graphs with girth at least 5 are $(1, 10)$-colorable

Jisu Jeong

KAIST

Joint work with Hojin Choi, Ilkyoo Choi, and Geewon Suh

October 31, 2014

DEFINITION

A graph G is properly k-colorable if the following is possible:

- color all vertices using k different colors
- no two adjacent vertices have the same color

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A graph G is $(d_1, d_2, \ldots, d_r)$-colorable if the following is possible:

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Observe that if a graph G is $(d_1, d_2, \dots, d_r)$-colorable,
then G is $(d_1 + 1, d_2, \dots, d_r)$-colorable.

EXAMPLE

- C_5 is not 2-colorable, that is, not $(0, 0)$-colorable.
- C_5 is $(0, 1)$-colorable.
- K_4 is not 3-colorable.
- K_4 is not $(0, 1)$-colorable.
- K_4 is $(1, 1)$-colorable.

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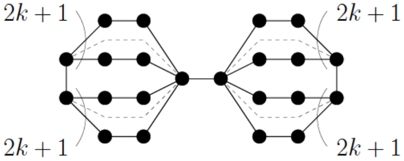
- K_4 is not $(0,1)$-colorable.
- K_4 is $(1,1)$-colorable.

Theorem (Borodin–Ivanova–Montassier–Ochem–Raspaud 2010)

The **girth** *of a graph is the length of a shortest cycle contained in the graph. For every k, there exists a planar graph with girth 6 that is not $(0,k)$-colorable.*



KNOWN RESULT

Theorem (Four Color Theorem; Appel–Haken 1977)

Every planar graph is $(0, 0, 0, 0)$*-colorable.*

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Every planar graph is $(0, 0, 0, 0)$-colorable.

Theorem (Cowen–Cowen–Woodall 1986)

Every planar graph is $(2, 2, 2)$-colorable.

Theorem (Eaton–Hull 1999, Škrekovski 1999)

For every k, there exists a non-$(1, k, k)$-colorable planar graph.

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Naturally, the next line of research is to consider (d_1, d_2)-coloring.

Theorem (Cowen–Cowen–Woodall 1986)

For every (d_1, d_2), there exists a non-(d_1, d_2)-colorable planar graph.

PROBLEM

Consider the girth condition!!
The **girth** of a graph is the length of a shortest cycle contained in the graph.

Question

Every planar graph with girth at least g is (d_1, d_2)-colorable.

Problem (1)

Given (d_1, d_2), determine the min $g = g(d_1, d_2)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

Problem (2)

Given $(g; d_1)$, determine the min $d_2 = d_2(g; d_1)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

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Problem (1)

Given (d_1, d_2), determine the min $g = g(d_1, d_2)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
1	10 or 11	6 or 7				
2	8	6 or 7	5 or 6			
3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5	5	5
6	7	5 or 6	5	5	5	5

- Every planar graph with girth at least 6 is $(1, 4)$-colorable.
- $\exists$ non-(d_1, d_2)-colorable planar graphs with girth 4 for all d_1,d_2.

KNOWN RESULT

Problem (2)

Given $(g; d_1)$, determine the min $d_2 = d_2(g; d_1)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

Theorem

For every g and d_1, it is known whether $d_2(g; d_1)$ exists or not, except $(g; d_1) = (5; 1)$.

girth	$(0, k)$	$(1, k)$	$(2, k)$	$(3, k)$	$(4, k)$
3	$\times$	$\times$	$\times$	$\times$	$\times$
4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$		$(2, 6)$	$(3, 5)$	$(4, 4)$
6	$\times$	$(1, 4)$	$(2, 2)$		
7	$(0, 4)$	$(1, 1)$			
8	$(0, 2)$				
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6	$\times$	$(1, 4)$	$(2, 2)$		
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Question (Montassier–Ochem 2014+)

Is there k where planar graphs with girth 5 are $(1, k)$-colorable?

MAIN THEOREM

Theorem (Choi–Choi–J.–Suh 2014+)

Every planar graph with girth at least 5 is $(1, 10)$-colorable.

$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
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2	8	6 or 7	5 or 6			
3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5	5	5
6	7	5 or 6	5	5	5	5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	7	5 or 6	5	5	5	5
11	7	5 or 6	5	5	5	5

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4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$	$(1,10)$	$(2, 6)$	$(3, 5)$	$(4, 4)$
6	$\times$	$(1, 4)$	$(2, 2)$		
7	$(0, 4)$	$(1, 1)$			
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MAIN THEOREM

Theorem (Choi–Choi–J.–Suh 2014+)

Every planar graph with girth at least 5 is $(1, 10)$*-colorable.*

Moreover, our proof extends to any surface instead of the plane.

Theorem (Choi–Choi–J.–Suh 2014+)

Given a surface S of Euler genus γ, every graph with girth at least 5 that is embeddable on S is $(1, K(\gamma))$*-colorable where* $K(\gamma) = \max\{10, 4\gamma + 3\}$.

FUTURE WORK

Question

Is there a planar graph with girth at least 5 that is not $(1,4)$-colorable?

Note that there is a planar graph with girth 5 that is not $(1,3)$-colorable.

Question

Is every planar graph with girth at least 5

- $(1,9)$*-colorable?*
- $(2,5)$*-colorable?*
- $(3,4)$*-colorable?*

Question

Is every planar graph with girth at least 6 $(1,3)$-colorable?


Thank you for your attention!